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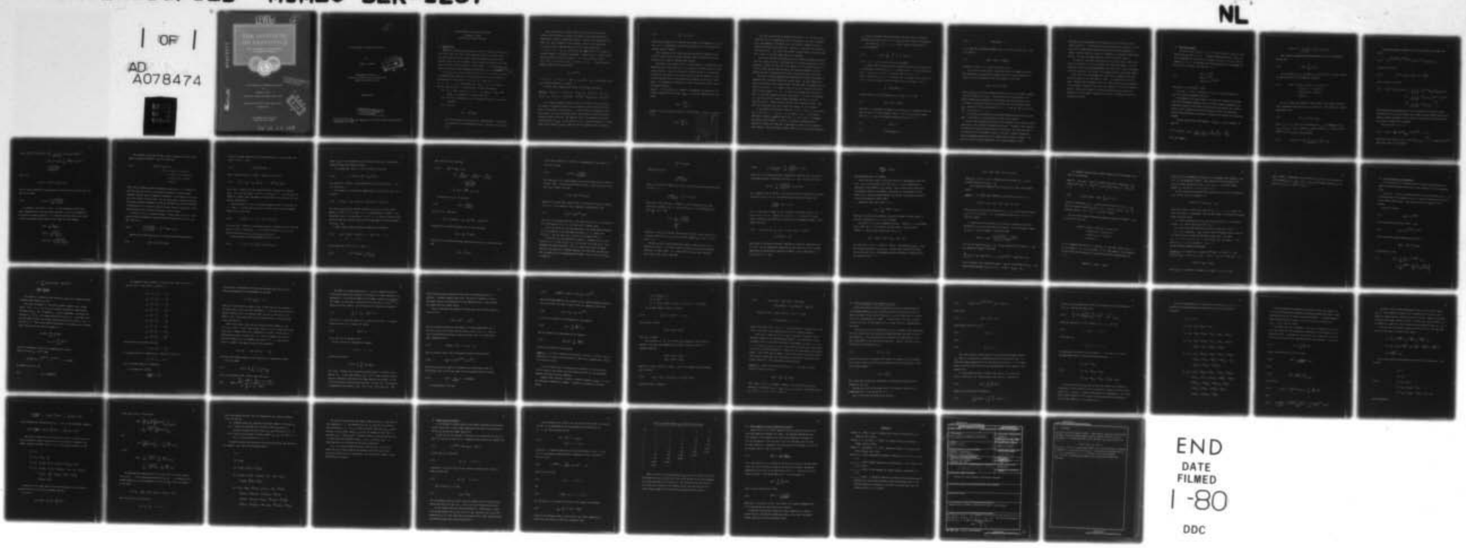

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On the Cumulants of Cumulative Processes

by

Walter L. Smith

University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series #1257

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DEPARTMENT OF STATISTICS
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On the Cumulants of Cumulative Processes

by Walter L. Smith
(University of North Carolina)

§1. Introduction.

Cumulative processes were introduced in 1955 as a natural concomitant to the study of what the author called regenerative processes (Smith, 1955). In that first paper some basic results concerning cumulative processes were obtained; these were also explained, and the notions of cumulative and regenerative processes further detailed, in Smith (1958). For the purpose of the present discussion ~~we shall introduce~~ the cumulative process ~~as follows.~~ IS PRESENTED.

Let $\{X_n\}_{n=1}^{\infty}$ be a *positive renewal process*, that is: an infinite sequence of independent and identically distributed random (iid) variables which are, with probability one, positive. Let us set $S_o = 0$ and, for $n = 1,2,\ldots$, $S_n = X_1 + X_2 + \ldots + X_n$. Then suppose $W(t)$, for $t \geq 0$, to be a stochastic process, such that $W(o) = 0$, satisfying the following requirements:

(C1) If $Y_1 = W(X_1)$, and, for $n = 2,3,\ldots$, $Y_n = W(S_{n+1}) - W(S_n)$, then $\{Y_n\}_{n=1}^{\infty}$ is a sequence of iid random variables.

(C2) $W(t)$ is, with probability one, of bounded variation in every finite t-interval.

(C3) If we set

$$\tilde{W}(t) = \int_{0-}^{t} |dW(u)|$$

for the associated "variation process" (defining $\tilde{W}(t)$ as identically zero should $W(t)$ be of unbounded variation), then $\tilde{W}(t)$ also satisfies (C1).

Such a process $W(t)$ is called a *cumulative process*, and since such processes were introduced they have been found to have wide applicability especially to problems in operations research. Typically $W(t)$ represents the total cost, fuel consumption, lost customers, and so on; the time points $\{S_n\}$ are so-called "regeneration points" of the underlying process.

For $r = 1,2,\ldots$, set $\nu_r = E(Y_1)^r$ when this moment exists (i.e. when the relevant integral is absolutely convergent). Similarly set $\tilde{\nu}_r = E(\tilde{Y}_1)^r$, where the implication of the tilde is obvious. Let us also write $\mu_r = E(X_1)^r$, $F(x) = P\{X_1 \leq x\}$ and $F_n(x) = P\{S_n \leq x\}$, for $n = 1,2,3,\ldots$. Finally, when the product moments exist, we shall write

$$\mu_{rs} = E\, X_1^r\, Y_1^s$$

$(r = 0,1,2,\ldots;\ s = 0,1,2,\ldots)$. Thus $\mu_r = \mu_{ro}$ and $\nu_s = \mu_{os}$. We similarly write $\tilde{\mu}_{rs}$ for product moments of X_1 and $\tilde{Y}_1$.

In Smith (1955), amongst other things, the following was proved:

<u>Theorem 1</u>. *Suppose* $\mu_2 < \infty$ *and* $\tilde{\nu}_2 < \infty$. *Let* $\sigma_1^2 = \mu_2 - \mu_1^2$, $\sigma_2^2 = \nu_2 - \nu_1^2$, *and* $\rho\sigma_1\sigma_2 = \mu_{11}$. *Then, as* $t \to \infty$, $\mathrm{Var}\, W(t) \sim \mu_1^{-1} t\{\sigma_2^2 - 2\rho\sigma_1\sigma_2(\nu_1/\mu_1) + \sigma_1^2(\nu_1/\mu_1)^2\}$.

A very special cumulative process is $N(t)$, the *renewal count*, which gives the number of regeneration points to have appeared in the interval $(0,t]$; thus $N(t)$ is the maximum k such that $S_k \leq t$. In Smith (1959) the cumulants of $N(t)$ were studied, and it was shown that under a weak condition on $F(k)$ and certain reasonable moment conditions, the cumulants of $N(t)$ are asymptotically linear. In more detail, if $\mu_{n+p+1} < \infty$ for some $n = 1,2,3,\ldots$ and some integer $p \geq 0$, then there exist constants a_n and b_n such that the n^{th} cumulant of $N(t)$ is given by

$$(1.1) \qquad a_n t + b_n + r_n(t),$$

where various things can be said about the rapidity of convergence of $r_n(t)$ to zero, as $t \to \infty$, depending on p and the assumption on $F(x)$. We refer to the original paper for details.

These cumulant results could have been substantially improved if the results of Smith (1966) on general remainder terms in renewal theory been available at the time the cumulants were studied.

The object of the present role is to extend Theorem 1 considerably and to demonstrate that results like (1.1) can be proved for certain classes of cumulative processes. However, a difficulty blocking the way of a very general treatment is that there is such a wide variety of possible behavior for $W(t)$ between regeneration points even when the distribution of the $\{Y_n\}$ is determined. To circumvent this difficulty we shall introduce two special classes of cumulative process:

<u>Class A</u>. Let $\{(X_n,Y_n)\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random vectors such that $\{X_n\}_{n=1}^{\infty}$ is a positive renewal process, and define

$$W(t) = \sum_{k=1}^{N(t)+1} Y_k .$$

<u>Class B</u>. Let $\{(X_n,Y_n)\}_{n=1}^{\infty}$ be as above, but define $W(t) = 0$ for $t < X_1$ and, for $t \geq X_1$,

$$W(t) = \sum_{k=1}^{N(t)} Y_k .$$

The Class A process can be thought of as follows. At zero time a cost is levied for the first regenerative cycle of the process, this cost Y_1 depends on X_1, the duration of that first cycle. At time X_1 a further cost Y_2 is levied for the next cycle, and so on. Then $W(t)$ is the total cost incurred in the interval $[0,t]$. The Class B only differs from the Class A in that one pays for a "cycle" at its end, rather than at its beginning.

The Class A and Class B processes pose very similar mathematical problems with very similar solutions. This paper will be devoted exclusively to Class A processes. However, at the end of this report, in §6, we briefly discuss the Class B process and show that the theory and formulae appropriate to that process are easily obtained from the theory provided for Class A processes.

The work is heavily dependent upon theorems which establish the nature of remainder terms in renewal theory and related topics. We shall depend very much on the results given in Smith (1966), and some words of explanation are in order at this point. It is possible that the $\{X_n\}$ are, almost surely, multiples of some constant $\tilde{\omega} > 0$. If is chosen to be maximal then we would say $W(t)$ is a *periodic* cumulative process, with period $\hat{\omega}$. If $W(t)$ is not periodic it is *aperiodic* and this report assumes throughout that $W(t)$ is such an aperiodic process. There would be a similar, and in some ways easier, treatment for the periodic case, but we shall not consider it in this report.

If the marginal distribution of the $\{X_n\}$ is $F(x) = P\{X_n \le x\}$, let us write $F_k(x) = P\{X_1 + X_2 + \ldots + X_k \le x\}$. If, for some $k = 1,2,\ldots$, $F_k(x)$ has an absolutely continuous component then we say F is in the class S. Throughout this paper we must assume $F \in S$; this is a technical assumption which is unlikely to present any difficulty in applying the results of this report to real problems. But the theorems of Smith (1966) are all based on the hypothesis

$F \in S$ and our statements about the rapidity with which various remainder functions of t tend to zero, as $t \to \infty$, would be false without this hypothesis.

Let us write $S_n = X_1 + \ldots + X_n$. Then we shall see below that the so-called ϕ-moments:

$$\phi_\ell(t) = \sum_{n=0}^{\infty} \binom{n + \ell - 1}{n} P\{S_n \leq t\}, \tag{1.2}$$

for $\ell = 1,2,\ldots$, play a crucial role in our arguments.

In Smith (1966) a useful class M^* of monotone functions M was introduced. A function $M(x)$, defined for $0 \leq x < \infty$, belongs to M if: it is non-decreasing; $M(x) \geq 1$ for all $x \geq 0$; $M(x + y) \leq M(x)M(y)$ for all $x \geq 0$, $y \geq 0$; $M(2x) = O(M(x))$ for all $x \geq 0$. A function $B(x)$, of bounded total variation on $(0-,\infty)$ is said to belong to the class $B(M;\nu)$, for some integer $\nu \geq 0$, if

$$\int_{0-}^{\infty} x^\nu M(x)\,|dB(x)| < \infty .$$

In Smith (1966) it is then shown that if $F \in B(M,\ell + 1) \cap S$ then

$$\phi_\ell(t) = P_\ell(t) + R_\ell(t) \tag{1.3}$$

where $P_\ell(t)$ is a polynomial of degree ℓ in t and $R_\ell(t)$ is a remainder term which tends to zero as $t \to \infty$ and belongs to $B(M;0)$. This is true for any integer value of $\ell \geq 1$. Thus the following statements are true, as $t \to \infty$,

$$R_\ell(t) \to 0 \tag{1.4}$$

$$\int^{\infty} M(u)\,|dR_\ell(u)| \to 0 \tag{1.5}$$

$$(1.6) \qquad M(t)\,\mathcal{R}_\ell(t) \to 0\ .$$

It is clear that a Chebyshev argument gives (1.6) from (1.5) and (1.4). Thus (1.3) implies

$$(1.7) \qquad \phi_\ell(t) = \mathcal{P}_\ell(t) + O\left(\frac{1}{M(t)}\right)\ ,$$

although, of course, (1.7) is much less than is true.

In §2 and §3 we shall prove, using these results on ϕ-moments that $W(t)$ does indeed have, asymptotically, cumulants which are linear functions of t. If $K_m(t)$ be the m^{th} cumulant of $W(t)$ then Theorem 3.1 will show

$$K_m(t) = A_m t + B_m + \mathcal{R}_m(t)$$

where A_m and B_m are constants and $\mathcal{R}_m(t)$ tends to zero at a rate shown to depend on the tail behavior of G and F. Ensuing sections are then devoted to the rather off-putting calculations of the constants $\{A_m\}$ and $\{B_m\}$ which are, as in advances, increasingly involved rational functions of the product moments $\mu_{\alpha\beta}$. Essentially the $\{A_m\}$ are given for $m = 1,\ldots,8$ and the $\{B_m\}$, for the case $8Y_j = 0$ only, for $m = 1,\ldots,6$.

In §5 checks are discussed of the detailed formulae given for the $\{A_m\}$ and $\{B_m\}$. In §6 there is a first discussion of the Class B process.

We close this introductory section with two comments on the reason for being interested in the cumulants of cumulative processes. In general, these processes are difficult to discuss theoretically with exactitude. But they arise widely in practice and it is desirable to have, at the least, reliable information about the sampling characteristics of sample means and sample variances. Thus one can easily be needing information on the fourth cumulants of $W(t)$

and even on the sixth cumulant if the skewness of the distribution of W(t) should interest us. A second reason for investigating these cumulants is that we may wish to approximate the distribution of W(t) with an Edgeworth, or similar, expansion based on successive refinements of a dominant normal approximation. The need for higher cumulants of W(t) then quickly appears.

Incidentally, the rather long formulae for A_n and B_n, when n grows large, are not as unwieldly as a first encounter with them might suggest. We have found that certain hand-held programmable calculators can be easily programmed to provide A_8 and B_6, for instance, for changing values of the product moments.

Lastly we wish to acknowledge that this work was triggered a few years ago by an enquiry by the late Mindel C. Sheps. In preparing her book with Jane A. Menken (Sheps and Menken, 1973), Dr. Sheps needed detailed information on the asymptotic behavior of the first few cumulants of a *periodic* cumulative process. Such results appear in the book in connection with interesting applications of the theory of cumulative processes to mathematical models of conception and birth.

§2. Some basic results.

Let us set $G(x,y) = P\{X_1 \leq x, Y_1 \leq y\}$ for the joint d.f. of (X_1,Y_1), and hence of (X_n,Y_n) for $n = 2,3,\ldots$. We have already written $F(x) = P\{X_1 \leq x\}$ for the marginal d.f. of X_1; let us therefore write $H(y) = P\{Y_1 \leq y\}$ for the marginal d.f. of Y_1. We introduce the following notation for transforms, in which s and θ are taken to be real and $s > 0$.

$$\begin{aligned} \mathcal{F}(s) &= E\, e^{-sX_1} \\ \mathcal{H}(\theta) &= E\, e^{i\theta Y_1} \\ \mathcal{G}(s,\theta) &= E\, e^{-sX_1+i\theta Y_1} \end{aligned} \tag{2.1}$$

Thus $\mathcal{G}(s,0) = \mathcal{F}(s)$ and $\mathcal{G}(0,\theta) = \mathcal{H}(\theta)$.

Let $\zeta_n(t) = 1$ if $S_n \leq t$ and $\zeta_n(t) = 0$ if $S_n > t$; we shall abbreviate $\zeta_n(t)$ to ζ_n where no ambiguity can arise.

Our argument will be much clearer if we deal with a specific case rather than giving the general argument; the way in which a general argument can be constructed should be clear from our discussion of the specific case, which we choose to be an examination of the fourth moment of $W(t)$; this particular problem being sufficiently non-trivial to bring out the features of the general argument.

A certain notation will prove helpful. If $p_1,p_2,\ldots,p_\ell$ are integers, we shall set

$$\textstyle\sum_t(p_1,p_2,\ldots,p_\ell) = \displaystyle\sum_{r_1>r_2>\ldots>r_\ell\geq 0}\sum\ldots\sum \zeta_{r_1} Y_{r_1+1}^{p_1} Y_{r_2+1}^{p_2} \ldots Y_{r_\ell+1}^{p_\ell} . \tag{2.2}$$

Thus, for example,

$$\sum_t(2,1,1) = \sum_{r_1>r_2>r_3\geq 0}\sum\sum \zeta_{r_1} Y^2_{r_1+1} Y_{r_2+1} Y_{r_3+1} .$$

Now it should be apparent from the definition of a Class A cumulative process that

(2.3) $$W(t) = \sum_{r=0}^{\infty} \zeta_r Y_{r+1} .$$

If we note that $\zeta_{r_1}\zeta_{r_2} = \zeta_{r_1}$ when $r_2 < r_1$ and that $\zeta_r^2 = \zeta_r$, then a careful calculation based on (2.3) yields the conclusion:

(2.4) $$\begin{aligned}\{W(t)\}^4 = &\sum_t(4) + 6 \sum_t(2,2) + 12 \sum_t(1,1,2) \\ &+ 12 \sum_t(2,1,1) + 12 \sum_t(1,2,1) \\ &+ 24 \sum_t(1,1,1,1) + 4 \sum_t(3,1) \\ &+ 4 \sum_t(1,3) .\end{aligned}$$

We wish to learn about $E\{W(t)\}^4$; thus we must be able to deal with individual expressions such that $E \sum_t(1,2,1)$, and so on. This leads us to consider expectations like

(2.5) $$E\, \zeta_{r_1} Y_{r_1+1} Y^2_{r_2+1} Y_{r_3+1}$$

where $r_1 > r_2 > r_3$. But it is to be noted that Y_{r_1+1} is independent of ζ_{r_1}, Y_{r_2+1}, and Y_{r_3+1}. This is because ζ_{r_1} depends only on S_{r_1}. Thus (2.5) simplifies to $\nu_1\, E\, \zeta_{r_1} Y^2_{r_2+1} Y_{r_3+1}$ and hence

(2.6) $$E \sum_t(1,2,1) = \nu_1 E \sum_{r_1>r_2>r_3\geq 0}\sum\sum \zeta_{r_1} Y^2_{r_2+1} Y_{r_3+1} .$$

In notation already introduced, if θ_2 and θ_3 represent real dummy variables,

$$(2.7)\qquad E\, e^{-sS_{r_1}+i\theta_2 Y_{r_2+1}+i\theta_3 Y_{r_3+1}} = \{F(s)\}^{r_1-2}\, G(s,\theta_2)\, G(s,\theta_3)\ .$$

Also let us note that

$$(2.8)\qquad \int_0^\infty e^{-st}\,\zeta_r(t)dt = \int_{S_r}^\infty e^{-st}dt = \frac{e^{-sS_r}}{s}.$$

Thus (2.8) and (2.6) yield

$$(2.9)\qquad \int_0^\infty e^{-st}\, E\, \textstyle\sum_t(1,2,1)dt = \nu_1 \displaystyle\sum_{r_1>r_2>r_3\geq 0}\sum\sum \int_0^\infty e^{-st}\, E\{\zeta_{r_1}(t)Y^2_{r_2+1}Y_{r_3+1}\}dt$$

$$= \nu_1 \sum_{r_1>r_2>r_3\geq 0}\sum\sum E\{Y^2_{r_2+1}Y_{r_3+1}\int_0^\infty e^{-st}\,\zeta_{r_1}(t)dt\}$$

$$= \frac{\nu_1}{s} \sum_{r_1>r_2>r_3\geq 0}\sum\sum E\{Y^2_{r_2+1}Y_{r_3+1}\, e^{-sS_{r_1}}\}\ .$$

The manipulations performed here are easily justified if we assume ν_2 exists (absolutely) for we can first replace the Y-terms by their absolute values, thus making all terms non-negative, and we shall see below that the resulting triple series must be (absolutely) convergent.

Let us set, for k = 1,2,...,

$$(2.10)\qquad C_k(s) = \left(\frac{\partial}{i\partial\theta}\right)^k G(s,\theta)\Big|_{\theta=0} = E\, Y_n^k\, e^{-sX_n},\quad n = 1,2,\ldots\ .$$

Then we see that, for $r_1 > r_2 > r_3 \geq 0$, $E\{Y^2_{r_2+1}\, Y_{r_3+1}\, e^{-sS_{r_1}}\} = C_2(s)C_1(s)\{F(s)\}^{r_1-2}$. This result, used in (2.9) yields,

$$(2.11) \quad \int_0^\infty e^{-st}\, E \textstyle\sum_t(1,2,1)dt = \frac{\nu_1}{s} \sum\sum\sum_{r_1>r_2>r_3\geq 0} C_1(s)\, C_2(s)\, \{F(s)\}^{r_1-2}$$

$$= \frac{\nu_1}{s}\, C_1(s)\, C_2(s) \sum_{r=2}^{\infty} \frac{r(r-1)}{2} \{F(s)\}^{r-2}$$

$$= \frac{\nu_1\, C_1(s)C_2(s)}{2s[1-F(s)]^3} .$$

Let us set

$$S(1,2,1) = \int_0^\infty e^{-st}\, E \textstyle\sum_t(1,2,1)dt,$$

with an obvious extension of this notation to terms like $S(3,1)$, $S(4)$, and so on. Thus (2.11) yields

$$(2.12) \qquad S(1,2,1) = \frac{\nu_1\, C_1(s)C_2(s)}{2s[1 - F(s)]^3} .$$

Incidentally, since $F(s) < 1$ for $s > 0$, the convergence of the triple series needed earlier to justify certain maneuvers is any easy consequence of the obvious finiteness of $S(1,2,1)$. It should be clear now how one can obtain a variety of similar results which we shall need; for some examples:

$$S(4) = \frac{\nu_4}{s[1 - F(s)]} ,$$

$$S(2,2) = \frac{\nu_2\, C_2(s)}{s[1 - F(s)]^2} ,$$

$$S(1,1,1,1) = \frac{\nu_1 [C_1(s)]^4}{3!\; s[1 - F(s)]^4} .$$

The relevance of all these S-terms is that, together, they give us the Laplace Transform of $E\{W(t)\}^4$. For (2.4) shows that

$$E\int_0^\infty e^{-st}\,\{W(t)\}^4\,dt \tag{2.13}$$
$$= S(4) + 6\,S(2,2) + 12\,S(1,1,2)$$
$$+ 12\,S(2,1,1) + 12\,S(1,2,1)$$
$$+ 24\,S(1,1,1,1) + 4\,S(3,1)$$
$$+ 4\,S(1,3).$$

Thus to gain information about the asymptotic behavior as $t \to \infty$ of $\{W(t)\}^4$ we must look upon each S-term as the Laplace Transform of a function whose asymptotic behavior must be separately determined. At this point we will make use of the results of Smith (1966). But the latter paper, essentially dealing with renewal theoretic problems in which the lifetimes $\{X_n\}$ are not restricted to being positive, is couched in terms of Fourier Steiltjes transforms; we must therefore make certain necessary adjustments to those results.

Consider $S(1,2,1)$ as a specific example. We may regard s $S(1,2,1)$ as the Laplace-Stieltjes transform of some non-decreasing function H(t; 1,2,1), say. Thus, from (2.1),

$$\frac{\nu_1\, C_1(s)C_2(s)}{2[1 - F(s)]^3} = \int_0^\infty e^{-st}\, H(dt;\, 1,2,1)\ . \tag{2.14}$$

Equation (2.10) shows that $C_k(s)$ is the Laplace-Stieltjes Transform of

$$C_k(x) = \int_{-\infty}^{+\infty} y^k\, G(x,dy)\ . \tag{2.15}$$

If $M(x)$ is a moment function of the kind introduced in §1 and we assume, for some $\ell = 0,1,2,\ldots$, that

$$E|X_1|^{\ell}|Y_1|^{k} M(X_1) < \infty, \tag{2.16}$$

then it follows that $C_k(x) \in \mathcal{B}(M;\ell)$. Moreover one can write

$$C_k(s) = \mu_{0k} - s\mu_{1k} + \frac{s^2}{2!}\mu_{2k} - \ldots + \frac{(-s)^{\ell}}{\ell!}\mu_{\ell k}\,\mathcal{D}_{\ell k}(s) \tag{2.17}$$

where $\mathcal{D}_k(s) \in \mathcal{B}(M;0)$ and is the Laplace-Stieltjes Transform of a function $D_{\ell k}(x)$, say, such that $D_{\ell k}(x) \equiv 0$ for $x < 0$, and $D_{\ell k}(\infty) = 1$. This follows from Theorem 3 of Smith (1966) and Lemma 4 of Smith (1959), with suitable, but fairly obvious, modifications.

With reference to (2.14), if we assume temporarily that (2.16) holds with $\ell = 3$ and $k = 2$, we have that $C_1(s)\ C_2(s) \in \mathcal{B}^*(M;3)$ and by the same argument behind (2.16) we can write

$$C_1(s)C_2(s) = \gamma_0 + s\gamma_1 + s^2\gamma_2 + s^3\gamma_3\,\mathcal{D}_0(s), \tag{2.18}$$

say, where $\mathcal{D}_0(s) \in \mathcal{B}^*(M;0)$ is the Laplace-Stieltjes Transform of $D_0(x)$ such that $D_0(x) \equiv 0$ for $x < 0$ and $D_0(\infty) = 1$. In (2.18) the constants $\gamma_0,\gamma_1,\gamma_2,\gamma_3$ are rational functions of $\mu_{01},\mu_{11},\mu_{21},\mu_{31},\mu_{02},\mu_{12},\mu_{22},\mu_{32}$.

On the other hand, the same arguments allow us to write

$$1 - F(s) = \mu_1 s - \frac{1}{2}\mu_2 s^2 + \frac{1}{6}\mu_3 s^3\,F_{(3)}(s)\ , \tag{2.19}$$

where $F_{(3)}(s)$ is the transform of the distribution function of some positive random variable, and belongs to $\mathcal{B}^*(M;0)$.

In a similar way, since $\{2 - F(s)\} \in \mathcal{B}^*(M;3)$, we can show

$$(2.20) \qquad \{1 - F(s)\}^2 = \mu_1^2 s^2 - \mu_2\mu_1 s^3 \mathcal{D}_1(s),$$

say, where $\mathcal{D}_1(s) \in \mathcal{B}(M;0)$, is the transform of $D_1(x)$ such that $D_1(x) \equiv 0$ for $x < 0$ and $D_1(\infty) = 1$.

If we combine (2.19) and (2.20) appropriately, we find from (2.18) that we can write

$$(2.21) \qquad C_1(s)C_2(s) = g_0 + g_1[1-F(s)] + g_2[1-F(s)]^2 + g_3 s^3 \mathcal{D}_2(s),$$

say, where g_0, g_1, g_2, g_3 are rational functions of μ_1, μ_2, μ_3, and the product moments μ_{rs} with $r = 0,1,2,3$ and $s = 1,2$. Furthermore $\mathcal{D}_2(s) \in \mathcal{B}^*(M;0)$ is the transform of $D_2(x)$, say, where $D_2(x) \equiv 0$ for $x < 0$. A careful calculation shows that $D_2(\infty)$ must be a rational function of the same moments and product moments as are $g_0, \ldots, g_3$.

In Smith (1959) certain convenient ϕ-moments were introduced:

$$(2.22) \qquad \phi_k(t) = E[N(t) + 1][N(t) + 2] \ldots [N(t) + k], \quad t \geq 0,$$
$$= 0 \qquad , \quad t < 0.$$

It was shown that, for $k = 1,2,\ldots$, and $s > 0$,

$$(2.23) \qquad \int_0^\infty e^{-st}\, \phi_k(dt) = \frac{k!}{[1 - F(s)]^k} \ .$$

But (2.14) and (2.21) show that

$$(2.24) \qquad \int_0^\infty e^{-st} H(dt;\, 1,2,1)$$

$$= \frac{\nu_1}{2} \quad \frac{g_0}{[1-F(s)]^3} + \frac{g_1}{[1-F(s)]^2} + \frac{g_2}{[1-F(s)]}$$

$$+ \frac{\nu_1 g_3 s^3 \mathcal{D}_2(s)}{2[1-F(s)]^3}$$

$$= \frac{\nu_1}{2} \Psi_1(s) + \frac{\nu_1 g_3}{2} \Psi_2(s), \text{ say.}$$

Consider first $\Psi_2(s)$. We can write

$$(2.25) \qquad \Psi_2(s) = \frac{K(s)}{[1-F(s)]^3} \quad ,$$

and, by (2.21), claim that

$$K(s) = C_1(s)C_2(s) - g_0 - g_1[1-F(s)] - g_2[1-F(s)]^2 .$$

Evidently $K(s) \in B^*(M;3)$ and those will be a $K(x)$ such that

$$K(s) = \int_{0-}^\infty e^{-sx} K(dx).$$

Thus $K(-i\theta)$ is the Fourier-Stieltjes transform of $K(x)$ and, in view of the fact that

$$K(s) = s^3 \mathcal{D}_2(s)$$

we can easily show $K(-i\theta) = O(|\theta|^3)$ in a neighborhood of the origin $\theta = 0$.

For real θ we have

$$\Psi_2(-i\theta) = \frac{K(-i\theta)}{[1-F(i\theta)]^3} . \tag{2.26}$$

This expression is in a form suitable for the direct application of Theorem 1 of Smith (1966). The conclusion of that theorem is that there exists a function $P_2(x)$ in $\mathcal{B}(M;0)$ such that

$$\Psi_2(-i\theta) = \int_{-\infty}^{+\infty} e^{i\theta x} P_2(dx) .$$

However, as we shall show a little later, the variation of $P_2(x)$ is confined to $[0-,\infty)$. If we assume this fact, temporarily without proof, then

$$\Psi_2(-i\theta) = \int_{0-}^{\infty} e^{i\theta x} P_2(dx) . \tag{2.27}$$

But $P_2(x)$ is of bounded variation, so the right-side of (2.27) is continuous in $I\theta \geq 0$ and analytic in $I\theta > 0$, if we now allow θ complex values.

But $F(-i\theta)$ and $K(-i\theta)$ are also continuous in the closed upper half-plane $I\theta \geq 0$ and analytic in the open upper half-plane $I\theta > 0$. Recall that $|F(-i\theta)| < 1$ for all real $\theta \neq 0$, in view of our assumption that $F \in$.

If $f_1(z)$ and $f_2(z)$ are functions of complex z, continuous in $Iz \geq 0$ and analytic in $Iz > 0$, and if $f_1(z) \equiv f_2(z)$ on the real axis, then $f_1(z) \equiv f_2(z)$ throughout the upper half-plane. This is a fairly easy result in complex analysis, best proved by a conformal mapping of the half-plane onto the unit disc and appealing to the Maximum Modulus Theorem. Thus, if the role of $f_1(z)$ is played by

$$\int_{0-}^{\infty} e^{izx} P_2(dx)$$

and that of $f_2(z)$ by

$$\frac{K(-iz)}{[1 - F(-z)]^3} ,$$

since (2.26) and 2.27) show $f_1(z) \equiv f_2(z)$ on the real axis, we may conclude that for real $s > 0$,

$$\frac{K(s)}{[1 - F(s)]^3} = \int_{0-}^{\infty} e^{-sx} P_2(dx) .$$

Thus we have shown that $\Psi_2(s)$, $\in B^*(M;0)$, is the transform of $P_2(x)$ whose variation is confined to $[0-,\infty)$. We can take $P_2(0-) = 0$ and determine $P_2(\infty)$ from $\lim_{s\to 0} \Psi_2(s)$. Thus

$$P_2(\infty) = \lim_{s\to 0} \frac{s^3 \mathcal{D}_2(s)}{[1 - F(s)]^3}$$

$$= \frac{\mathcal{D}_2(0)}{\mu_1^3} .$$

Therefore, in view of our earlier consideration of $\mathcal{D}_2(0)$, we may claim $P_2(\infty)$ is a rational function of μ_1,μ_2,μ_3, and the product moments μ_{rs}, with $r = 0,1,2,3$, and $s = 1,2$.

We must now cover a point glossed over earlier, that $P_2(x)$ has all its variation in $[0-,\infty)$. To do this we refer to Theorems 4 and 5 and the ensuing discussion, in Smith (1966). It is shown there that there exist constants $A_1(4)$, $A_2(4)$, $A_3(4)$, $A_4(4)$, such that

$$(2.28) \qquad \frac{1}{\{1 - F(-i\theta)\}^3} = \sum_{j=1}^{4} \frac{A_j(4)}{(-\mu_1 i\theta)^{4-j}} + \Lambda(\theta),$$

where $\Lambda(\theta)$ is the Fourier-Stieltjes Transform of a function $L(x)$, say, which is of bounded variation, identically zero for $x < 0$. But (2.26) and (2.28) give

$$\Psi_2(-i\theta) = \sum_{j=1}^{4} \frac{A_j(4)K(-i\theta)}{(-\mu_1 i\theta)^{4-j}} + K(-i\theta)\,\Lambda(\theta)\ .$$

It is immediate that $K(-i\theta)\Lambda(\theta)$ is the Fourier-Stieltjes Transform of a function identically zero in $(-\infty,0)$. We therefore need only deal with terms like

$$\frac{K(-i\theta)}{C(-i\theta)^r} \quad \text{for } r = 1,2,3.$$

For $r = 3$ this term is simply $\mathcal{D}_2(-i\theta)$, and there is nothing to prove. If $r < 3$ we must turn to (2.21). Suppose, for instance, $r = 2$. Theorem 3 of Smith (1966), especially the proof of this result and the accompanying Lemma 1, shows that we can write

$$(2.29) \qquad C_1(-i\theta)C_2(-i\theta) - g_0 - g_1[1 - F(-i\theta)] - g_2[1 - F(-i\theta)]^2$$

$$= (-i\theta)^2\ \mathcal{D}_3(\theta)\ , \text{ say,}$$

where $\mathcal{D}_3(\theta)$ is the Fourier-Stieltjes Transform of a function, identically zero in $(-\infty,0)$. This is obtained by expressing the left-side of (2.29) as a linear combination of characteristic functions in $B^{\neq}(M;1)$, and by noting that it is $O(|\theta|^3)$ near $\theta = 0$. Thus

$$\frac{K(i\theta)}{(-i\theta)^2} = \mathcal{D}_3(\theta),$$

and the desired result is obtained.

We must now hark back to (2.24) and discuss the significance of the term $\Psi_1(s)$. But it is plain from (2.23) that $\Psi_1(s)$ is a linear combination of transforms of the ϕ-moments ϕ_1, ϕ_2, and ϕ_3. Thus we merely need to know about the asymptotic behavior of ϕ-moments; this matter was investigated in Smith (1959) but, as we have already remarked, more general conclusions are possible if we use the results of Smith (1966).

Computation shows that, for $k = 1,2,\ldots$,

$$\phi_k(t) = \sum_{n=0}^{\infty} \binom{n+k-1}{n} P\{S_n \le t\} .$$

Thus $\phi_k(t)$ is precisely the function covered by Theorem 5 of Smith (1966), in which the substitution $\ell = k + L$ is to be made.

The cases $k = 1,2,3$ are of immediate interest. Consider $k = 3$, for example. Here we must take $\ell = 4$ in Theorem 5 of Smith (1966). We must assume $E\, X^4 M(X) < \infty$, and can then conclude that, for $t > 0$,

$$\phi_3(t) = A_{33}t^3 + A_{32}t^2 + A_{31}t + A_{30} + \Lambda(t),$$

say, where $\Lambda(t) \to 0$ as $t \to \infty$ and $\Lambda(t) \in \mathcal{D}(M;0)$. The constants $A_{33}, A_{32}, \ldots, A_{30}$ are rational functions of $\mu_1, \mu_2, \mu_3, \mu_4$. Similar results hold for ϕ_2 and ϕ_1. Thus we see that $\Psi_1(s)$ is the Laplace-Stieltjes Transform of a function $P_1(t)$, say, such that for $t > 0$:

$$P_1(t) = \Pi_3 t^3 + \Pi_2 t^2 + \Pi_1 t + \Pi_0 + p(t),$$

where $p(t) \to 0$ as $t \to \infty$, $p(t) \in \mathcal{B}(M;0)$, and the constants $\Pi_0,\ldots,\Pi_4$ are rational functions of $\mu_1,\mu_2,\ldots,\mu_4$.

If we combine our findings on $\Psi_1(s)$ and $\Psi_2(s)$ we obtain the following result.

<u>Lemma 2.1.</u> *If* $E\,X^4 M(X)$ *and* $E\,X^3 Y^2 M(X)$ *are both finite, then for* $t > 0$,

$$\nu_1^{-1} H(t;\, 1,2,1) = k_3 t^3 + k_2 t^2 + k_1 t + k_0 + p(t),$$

where k_0,k_1,k_2,k_3 *are rational functions of* $\mu_1,\ldots,\mu_4$, *and the product moments* μ_{rs} *for* $r = 0,1,2,3$ *and* $s = 1,2$. *The remainder function* $p(t) \to 0$ *as* $t \to \infty$, *and* $p(t) \in \mathcal{B}(M;0)$.

It should not be too difficult to see that the treatment we have just afforded $S(1,2,1)$ can be given any of the S terms. For a more general example, let $p_1,p_2,\ldots,p_k$ be k integers. Then one can show

$$S(p_1,p_2,\ldots,p_k) = \frac{\nu_{p_1} C_{p_2}(s) C_{p_3}(s) \ldots C_{p_k}(s)}{(k-1)!\,[1 - F(s)]^R}\,.$$

Let q be the largest of $p_2,p_3,\ldots,p_k$. We must assume $E\,X_1^k |Y_1|^q M(X_1) < \infty$ and can expand the C-product as follows:

$$\prod_{j=2}^{k} C_{p_j}(s) = g_0 + g_1[1-F(s)] + \ldots + g_{k-1}[1-F(s)]^{k-1} + g_k s^k \mathcal{D}_k(s), \text{ say.}$$

In this expansion, the coefficients $g_0,g_1,\ldots,g_k$ will be functions of $\mu_1,\mu_2,\ldots,\mu_k$ and the product moments μ_{rs} for $r = 0,1,\ldots,k$ and $s = p_2,p_3,\ldots,p_k$.

The argument then goes much as before, making use of the ϕ-moments, and leads to the following.

Lemma 2.2. *Let* $p_1, p_2, \ldots, p_k$ *be* k *integers and write* $q = \max(p_2, p_3, \ldots, p_k)$. *Assume* $E\, X_1^{k+1} M(X_1) < \infty$, $E|Y_1|^{p_1} < \infty$, *and* $E\, X_1^k |Y_1|^q M(X_1) < \infty$. *Then, for* $t > 0$,

$$\nu_{p_1}^{-1} H(t;\, p_1, p_2, \ldots, p_k) = \sum_{j=0}^{k} c_j t^j + \rho(t)\,, \tag{2.30}$$

where the coefficients $c_0, c_1, \ldots, c_k$ *are rational functions of* $\mu_1, \mu_2, \ldots, \mu_{k+1}$ *and of the product moments* μ_{rs} *with* $r = 0, 1, \ldots, R$ *and* $s = p_2, p_3, \ldots, p_k$. *The remainder function* $\rho(t) \to 0$ *as* $t \to \infty$ *and* $p(t) \in B(M;0)$.

We can use this lemma to deduce the asymptotic behavior of $E\{W(t)\}^4$. From (2.12) we can infer that

$$\begin{aligned} E\{W(t)\}^4 &= H(t;4) + 6\, H(t;2,2) + 12\, H(t;1,1,2) \\ &\quad + 12\, H(t;2,1,1) + 12\, H(t;1,2,1) \\ &\quad + 24\, H(t;1,1,1,1) + 4\, H(t;3,1) \\ &\quad + 4\, H(t;1,3)\,. \end{aligned}$$

If it is supposed that $E|Y_1|^4 < \infty$, $E\, X_1^5 M(X_1) < \infty$, and that $E\, X_1^r |Y_1|^s M(X_1) < \infty$ for all integer $r \leq 4$, $s \leq 3$, such that $r + s \leq 5$, then Lemma 2.2 can be applied to each of the above H-functions in turn and the results establish that, for $t > 0$,

$$E\{W(t)\}^4 = P_4(t) + R_4(t)$$

where $P_4(t)$ is a polynomial in t and $R_4(t)$ is a remainder term, tending to zero as $t \to \infty$, and belonging to $B(M;0)$. The coefficients of the polynomial $P(t)$ will be rational functions of $\mu_1,\mu_2,\ldots,\mu_5$, $\nu_1,\nu_2,\ldots,\nu_4$, and the product moments μ_{rs} such that $r \leq 4$, $s \leq 3$, and $r + s \leq 5$.

We can evidently anticipate at this stage what will be true in the general case; we shall have, for integer $m \geq 1$,

$$E\{W(t)\}^m = \sum^* H(t;p_1,p_2,\ldots,p_R)$$

where the $\sum^*$ means a sum over all partitions $p_1 + p_2 + \ldots + p_R = m$, where the order of the parts is significant. When we apply Lemma 2.2 we find the following moment conditions arise:

(i) Since one partition of m is into the sum of m units, it is possible to have $k = m$. Thus we must have $E\, X_1^{m+1}\, M(X_1) < \infty$.

(ii) Since the H-function $H(t;m)$ can arise, we must have $E|Y_1|^m < \infty$.

(iii) If $q = \max(p_2,p_3,\ldots,p_k)$ is given, then the maximum possible value for k is when all the p's except the maximum one equal unity. This makes $(k - 1) + q = m$, i.e. $k + q = m + 1$. Thus we need $E\, X_1^k\, |Y_1|^q\, M(X_1) < \infty$ for every $q \leq m - 1$, $k \leq m$, such that $k + q \leq m + 1$.

We have therefore deduced the following.

<u>Theorem 2.3</u>. *For integer* $m \geq 1$, *assume* $E\, X_1^{m+1}\, M(X_1) < \infty$, $E|Y_1|^m < \infty$, *and* $E\, X_1^r |Y_1|^s\, M(X_1) < \infty$ *for* $r \leq m$; $s \leq m - 1$, *and* $r + s \leq m + 1$. *Then, for* $t > 0$,

$$E\, W(t)^m = P_m(t) + R_m(t),$$

where $P_m(t)$ *is a polynomial of degree* m *in* t, $R_m(t) \to 0$ *as* $t \to \infty$, *and*

$R_m(t) \in B(M,0)$. *Furthermore, the coefficients in* $P_m(t)$ *are rational functions of the moments* $\mu_1, \mu_2, \ldots, \mu_{m+1}$, $\nu_1, \nu_2, \ldots, \nu_m$, *and the product moments* μ_{rs} *such that* $r \leq m$, $s \leq m - 1$, *and* $r + s \leq m + 1$.

§3. On the linearity of the cumulants.

In this section we are concerned with the actual rational functions of moments and product moments which arise as the coefficients in the polynomials $P_m(t)$.

If $g(x_1,x_2,\ldots,x_m)$ be a rational function of the real variables $x_1,x_2,\ldots,x_m$ in Euclidean space of m dimensions then it is uniquely determined by its values in any sphere, however small. This idea motivates the argument of this section.

For real θ, define

$$M_\theta(t) = E\, e^{i\theta W(t)} \tag{3.1}$$

as the characteristic function of $W(t)$. Thus

$$M_\theta(t) = E\, e^{i\theta\Sigma_0^\infty Y_{j+1}\,\zeta_j(t)} . \tag{3.2}$$

If we calculate the Laplace Transform, for real $s > 0$,

$$M_\theta^0(s) = \int_0^\infty e^{-st}\, M_\theta(t)dt ,$$

we find from (3.2) that

$$\begin{aligned} M_\theta^0 &= E \sum_{\ell=0}^{\infty} \int_{S_\ell}^{S_{\ell+1}} e^{-st+i\theta\Sigma_1^{\ell+1} Y_j}\, dt \\ &= E \sum_{\ell=0}^{\infty} e^{i\theta\Sigma_1^{\ell+1} Y_j} \left\{ \frac{e^{-sS_\ell} - e^{-sS_{\ell+1}}}{s} \right\} \end{aligned} \tag{3.3}$$

$$= s^{-1} \sum_{\ell=0}^{\infty} \{[G(s,\theta)]^{\ell} H(\theta) - [G(s,\theta)]^{\ell+1}\}$$

$$= \frac{H(\theta) - G(s,\theta)}{s[1 - G(s,\theta)]} .$$

This formula is essentially that derived by a much less transparent method in Smith (1955, equation (5.2.11)).

Let us fix an integer $k \geq 1$ and choose another integer m much larger than k. For $j = 1,2,\ldots,m$, let X_j be a negative exponential random variable with mean value λ_j, say. We assume $X_1,\ldots,X_m$ are independent. Now define new variables $Y_j = \lambda_j^k X_j$. Let a random selection of the $X_1,X_2,\ldots,X_m$ be made, each X_j being equally likely, and let the resulting selection be called X. If $X_j = X$ then set $Y_j = Y$. Thus we have defined the probability distribution of the random vector (X,Y), and we see that for integer r,s: -

$$m\,E\,X^r Y^s = \sum_{j=1}^{m} \lambda_j^{ks} E\,X_j^{r+s}$$

$$= (r+s)! \sum_{j=1}^{m} \lambda_j^{ks+r+s} .$$

It will be convenient to say (X,Y) is generated by a λ-model.
Hence, if we set $\mu_{rs} = E\,X^r Y^s$, then

$$\frac{m}{(r+s)!}\mu_{rs} = \lambda_1^{r+(k+1)s} + \lambda_2^{r+(k+1)s} + \ldots \text{ (m terms)} .$$

For simplicity, let us set

$$\zeta_{rs} = \frac{m}{(r+s)!}\mu_{rs} \tag{3.4}$$

The argument we wish to pursue is clearer at this stage if we look at a specific case. If we choose k = 3 and m = 15

$$\zeta_{10} = \lambda_1 + \lambda_2 + \dots + \lambda_{15}$$

$$\zeta_{20} = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_{15}^2$$

$$\zeta_{30} = \lambda_1^3 + \lambda_2^3 + \dots + \lambda_{15}^3$$

$$\zeta_{01} = \lambda_1^4 + \lambda_2^4 + \dots + \lambda_{15}^4$$

$$\zeta_{11} = \lambda_1^5 + \lambda_2^5 + \dots + \lambda_{15}^5$$

$$\zeta_{21} = \lambda_1^6 + \lambda_2^6 + \dots + \lambda_{15}^6$$

$$\zeta_{31} = \lambda_1^7 + \lambda_2^7 + \dots + \lambda_{15}^7$$

$$\zeta_{02} = \lambda_1^8 + \lambda_2^8 + \dots + \lambda_{15}^8$$

$$\zeta_{33} = \lambda_1^{15} + \lambda_2^{15} + \dots + \lambda_{15}^{15}$$

These equations establishes the vector

$$\underline{\zeta} = (\zeta_{10}, \zeta_{20}, \dots, \zeta_{33})$$

in Euclidean Space of 15 dimensions as a function of the vector

$$\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{15}) ,$$

also in Euclidean space of 15 dimensions.

If we compute the Jacobian

$$\left|\frac{\partial(\underline{\zeta})}{\partial(\underline{\zeta})}\right| = J, \text{ say,}$$

we are led to a determinant of the familiar Vandermond type, and by a well-known representation of that determinant we find that

$$J = (15!) \left| \prod_{r>s} (\lambda_r - \lambda_s) \right| .$$

Thus if $\underline{\lambda}$ be restricted to a region D, say, in which none of its coordinates are equal, then J will be non-zero throughout D. It follows from the theory of implicit functions (see, e.g. Goursat (1904), pp. 45-51) that there will be sphere E, say, in the $\underline{\zeta}$-space and a vector-valued function $\underline{\lambda} = \underline{\lambda}(\underline{\zeta})$ which maps E into some part of D.

What we have seen is that if we are given the product moments μ_{rs} for $f = 0,1,2,3$ and $s = 0,1,2,3$, and if these lead via (3.4) to a point $\underline{\zeta}$ in E then there exists a random vector (X,Y), generated by a λ-model. Plainly there is nothing special about our choice of $k = 3$. The above argument will work for arbitrary integers $k \geq 1$; we evidently must take $m = k^2 - 1$. Thus, given k, there is a sphere in the space of vectors with coordinates:

$$(\mu_{10}, \mu_{20}, \ldots, \mu_{k0}, \mu_{11}, \mu_{21}, \ldots, \mu_{kk})$$

such that these product moments are all generated by an appropriate λ-model.

For such a model

$$G(s,\theta) = \frac{1}{m} \sum_{j=1}^{m} \frac{1}{1 + s\lambda_j - i\theta\lambda_j^k} . \tag{3.5}$$

If we use the equation $H(\theta) = G(0,\theta)$, then (3.3) gives

$$M_\theta^0(s) = \frac{\sum_{j=1}^{m} (1 - i\theta\lambda_j^k)^{-1} - \sum_{j=1}^{m} (1 + s\lambda_j - i\theta\lambda_j^k)^{-1}}{m - \sum_{j=1}^{m} (1 + s\lambda_j - i\theta\lambda_j^k)^{-1}} \tag{3.6}$$

Thus $M_\theta^0(s)$ is a rational function of s, it may be expanded in partial fractions and then easily inverted to yield $M_\theta(t)$ as a linear combination of exponentials. If the poles of $M_\theta^0(s)$ in the complex s-plane are all distinct, for example, we would have an expression of the form $M_\theta(t) = \sum \Psi_\nu e^{z_\nu(\theta)t}$ in which the critical numbers $z_\nu(\theta)$ are roots of the characteristic equation

$$(3.7) \qquad \sum_{j=1}^{m} (1 + z_\nu\lambda_j - i\theta\lambda_j^k)^{-1} = m .$$

From this it is clear that there will be exactly m such roots. It might be noted also that (3.7) is exactly the same as

$$(3.8) \qquad G[z,\theta] = 1,$$

a fact that will be important later.

We may, with no loss of generality, suppose

$$\lambda_1 > \lambda_2 > \dots > \lambda_n .$$

Consider the function

$$G(z,0) = \frac{1}{m} \sum_{j=1}^{m} \frac{1}{(1 + z\lambda_j)} ,$$

for z real. Evidently $G(z,0)$ increases without bound as z decreases from 0 towards $-\lambda_1^{-1}$. In the open interval $(-\lambda_2^{-1}, -\lambda_1^{-1})$ it again increases continuously as z decreases, from arbitrarily large negative values when z is near $-\lambda_1^{-1}$ to arbitrarily large positive values when z is near $-\lambda_2^{-1}$. This behavior is repeated for the intervals $(-\lambda_3^{-1}, -\lambda_2^{-1})$, $(-\lambda_4^{-1}, -\lambda_3^{-1})$, and so on. Thus

each of these intervals contains one real root of the equation $G(z,0) = 1$, giving m - 1 distinct negative real roots. But there is obviously a root at the origin, and so all possible roots of the equation $G(z,0) = 1$ are accounted for (there can be no complex roots).

Thus, by the theory of algebraic functions there are m distinct functions, roots of (3.7),

$$z_1(\theta),\ z_2(\theta),\ \ldots,\ z_m(\theta)$$

which are analytic functions of the complex θ in some neighborhood of $\theta = 0$. Let $z_1(\theta)$ be that one of these functions which satisfies $z_1(0) = 0$. Then, in view of the preceding discussion, we may claim that, for θ in a sufficiently small neighborhood of 0,

$$\text{(3.9)} \qquad Rzj(\theta) < -\lambda_2^{-1}, \quad j = 2,3,\ldots,m.$$

Thus we can write, with a nod to the partial fraction discussion above,

$$\text{(3.10)} \qquad M_\theta(t) = Z_1(\theta)e^{tz_1(\theta)} + O(e^{-t\lambda_2^{-1}}),$$

where the O-term can be shown to be uniform for all sufficiently small $|\theta|$. The function $Z_1(\theta)$, by the usual logic, can be calculated from the relation

$$\text{(3.11)} \qquad Z_1(\theta) = \lim_{z \to z_1(\theta)} (z - z_1(\theta))M_\theta^0(z).$$

It follows from (3.10) that

$$\log M_\theta^0(t) = z_1(\theta).t + \log Z_1(\theta) + O(e^{-t\lambda_2^{-1}}) \, . \tag{3.12}$$

But the left-hand member of this equation is the cumulant generating function, and we see from (3.12) that, if $K_r(t)$ be the r-th cumulant of $W(t)$, then

$$K_r(t) = A_r t + B_r + O(e^{-t\lambda_2^{-1}}) \, . \tag{3.19}$$

In (3.19) the constants A_r are determined by the expansion

$$z_1(\theta) = \sum_{r=1}^{\infty} \frac{(i\theta)^r}{r!} A_r \, , \tag{3.20}$$

and the constants B_r are determined by the expansion

$$\log Z_1(\theta) = \sum_{r=1}^{\infty} \frac{(i\theta)}{r!} B_r \tag{3.21}$$

We have thus established the following.

<u>Lemma 3.1</u>. *If a class* A *cumulative process* $W(t)$ *is based on a sequence* $\{(X_n, Y_n)\}$ *generated by a* λ*-model, then the cumulants of* $W(t)$ *are all asymptotically linear, in the sense of* (3.19).

Let us use this result in conjunction with Theorem 2.3. For clarity we shall be specific and consider the third cumulant of $W(t)$; the way the general argument would go will then be apparent.

Suppose, then, that $W(t)$ is based on a general sequence $\{(X_n, Y_n)\}$, i.e. one not necessarily produced by a λ-model. In order to use Thoerem 2.3 we shall assume;

(a) $E\, X_1^4\, M(X_1) < \infty$;

(b) $E\, |Y_1|^3 \quad < \infty$;

(c) $E \quad X_1^r\, |Y_1|^s\, M(X_1) < \infty$, for $r = 0,1,2,3,$; $s = 0,1,2, r+s\leq 4$.

Let us denote moments of $W(t)$ as follows:

$$(3.22) \qquad M_k(t) = E\, \{W(t)\}^k\ , \quad k = 1,2,\ldots\ .$$

Then Theorem 2.3 gives

$$M_3(t) = P_3(t) + R_3(t),$$

where $R_3(t) \in B(M;0)$.

But conditions (a), (b), (c) are more than adequate to deal with $M_2(t)$ and $M_1(t)$, and they yield progressively stronger conclusions about the remainder functions: -

$$M_2(t) = P_2(t) + R_2(t)$$
$$M_1(t) = P_1(t) + R_1(t)$$

where $R_2(t) \in B(M;1)$ and $R_1(t) \in B(M;2)$. Thus, if we compute the third cumulant $K_3(t)$, we obtain

$$(3.21) \qquad K_3(t) = P_3(t) - 3P_2(t)P_1(t) + 2[P_1(t)]^3 + R(t)\ .$$

The function $R(t)$ is given by

$$(3.22) \qquad R(t) = R_3(t) - 3R_2(t)P_1(t) - 3P_2(t)R_1(t)$$
$$+ 6[P_1(t)]^2R_1(t) + 6P_1(t)[R_1(t)]^2 + 2[R_1(t)]^3$$

Note that if $f(t) \in B(M;r)$ and $f(t) \to 0$ as $t \to \infty$, then

$$\int_t^\infty M(u)u^r|df(u)| \to 0, \quad t \to \infty,$$

implies that $M(t)t^r f(t) \to 0$ as $t \to \infty$. This fact makes it a quick task to show that all the terms on the right-hand side of (3.22) belong to $B(M;0)$.

Thus $K_3(t)$ is shown by (3.21) to equal a polynomial of degree 3 plus a remainder tending to zero and belonging to $B(M;0)$. The coefficients of this polynomial must all be rational functions of the moments μ_{rs} with $r \leq 3$, $s \leq 2$, $r + s \leq 4$ and μ_1,μ_2,μ_3,μ_4, ν_1,ν_2,ν_3. These rational functions will be the same if the $\{(X_n,Y_n)\}$ were generated by a λ-model with the same marginal and product-moments. But we have seen in Lemma 3.1 that in the latter case all the coefficients must be identically zero except for the coefficients of the first and zeroth. powers of t. Thus we have the following result.

<u>Theorem 3.1</u>. *Assume the conditions of Theorem 2.1. Then* $K_m(t)$, *the mth. cumulant of* $W(t)$, *is given by*

$$K_m(t) = A_m t + B_m + R_m(t)$$

where $R_m(t) \to 0$ *as* $t \to \infty$ *and* $R_m(t) \in B(M;0)$. *The constants* A_m *and* B_m *are functions of the moments* $\mu_1,\mu_2,\ldots,\mu_{m+1}$, $\nu_1,\nu_2,\ldots,\nu_m$ *and all the product-moments* μ_{rs} *with* $r \leq m$, $s \leq m - 1$, $r + s \leq m + 1$.

§4. On the calculation of the constants A_n and B_n .

We have seen that the desired constants A_n and B_n are rational functions of the product-moments μ_{rs} and it follows from the discussion of §3 that, in order to determine these rational functions, it will be sufficient if we assume the $\{Y_n\}$ to be generated by a λ-model. It follows from (3.12) that, in order to obtain the $\{A_n\}$, we need expand $z_1(\theta)$ as a power series in a neighborhood of the origin.

For low values of n, the coefficients $\{A_n\}$ are reasonably simple expressions; but as n increases the $\{A_n\}$ become increasingly cumbersome functions of the product moments μ_{rs}. Thus any ruse which will diminish the complication is desirable; unfortunately, few such ruses seem available. However, one that is available involves replacing the $\{Y_n\}$ by

$$\tilde{Y}_n = Y_n - \nu_1 X_n . \tag{4.1}$$

The use of these $\{\tilde{Y}_n\}$ will substantially reduce the complexity of our formulae. But it must be borne in mind that these formulae will now be expressed in terms of the product moments

$$\tilde{\mu}_{rs} = E\, X_n^r \tilde{Y}_n^s .$$

Thus a price must be paid for some degree of simplification achieved in the formulae for the $\{A_n\}$.

We know that $z_1(\theta)$ is the unique root of (3.8) which is analytic in a neighborhood of $\theta = 0$ and vanishes at $\theta = 0$.

But, in an obvious extension of our notation,

(4.2) $$\tilde{G}(z,\theta) = E\, e^{-zX+i\theta Y-i\nu_1\theta X} = G(z + i\nu_1\theta, \theta) .$$

Thus we see

(4.3) $$\tilde{z}_1(\theta) = z_1(\theta) - i\nu_1\theta ,$$

which implies that for all $n \geq 2$

(4.4) $$\tilde{A}_n = A_n ,$$

while

(4.5) $$\tilde{A}_1 = A_1 - \nu_1 .$$

Thus, apart from the tedious necessity of calculating the product moments $\tilde{\mu}_{rs}$, there is no difficulty in obtaining the desired $\{A_n\}$ from the $\{\tilde{A}_n\}$. To avoid the frequent appearance of the tilde we may now assume $\nu_1 = 0$; we comment again at the end of this section on the appropriateness of our formulae to the general case.

It is well known, and easy to deduce, that when $\nu_1 = 0$ we must have $E\, W(t) = 0$ for all $t \geq 0$. Thus we have at once that $A_1 = 0$ and may set

(4.6) $$z_1(\theta) = \sum_{k=2}^{\infty} \frac{A_k}{k!} (i\theta)^k .$$

Equation (3.8) then yields the identity

(4.7) $$\sum_{n=1}^{\infty} \frac{1}{n!} E(i\theta Y - X \sum_{k=2}^{\infty} \frac{A_k}{k!} (i\theta)^k)^n = 0 .$$

Let $\theta r/r!$ be the coefficient of $(i\theta)^r$ in the expression on the left-hand side of (4.7). Then one can deduce that

$$\text{(4.8)} \qquad \frac{\theta_r}{r!} = \sum^* \frac{(-1)^{\rho_2+\rho_3+\ldots+\rho_s}}{\rho_1!\rho_2!\ldots\rho_s!} \left(\frac{A_2}{2!}\right)^{\rho_2} \ldots \left(\frac{A_s}{s!}\right)^{\rho_s} \mu_{n-\rho_1,\rho_1}$$

where the summation $\sum^*$ is over integers $\rho_1,\rho_2,\ldots,\rho_s$, such that

$$\text{(4.9)} \qquad \rho_1 + 2\rho_2 + \ldots + s\rho_s = r$$

and we have set

$$\rho_1 + \rho_2 + \ldots + \rho_s = n .$$

By examining all partitions of the integers 1,...,8 in turn it is a fairly straightforward matter to obtain the following results.

$$\theta_2 = \mu_{02} - A_2\mu_{10} .$$

$$\text{(4.10)} \qquad \theta_3 = \mu_{03} - 3A_2\mu_{11} + A_3\mu_{10}$$

$$\theta_4 = \mu_{04} - 6A_2\mu_{12} + 4A_3\mu_{11} + 3A_2^2\mu_{20} - A_4\mu_{10} .$$

We also obtain $\theta_5,\theta_6,\theta_7,\theta_8$, but will not quote them in the interest of saving space. At this point it is also convenient to adjust the X and Y scales so that $\mu_{10} = 1$ and $\mu_{02} = 1$ (bear in mind that we are supposing $\mu_{01} = 0$). Since every θ_r must vanish, by (4.7), we are led in particular from the equations (4.10) to the result $A_2 = 1$, by setting $\theta_2 = 0$.

If we set the remaining θ's to zero and use the values $\mu_{10} = \mu_{02} = A_2 = 1$ we obtain the following equations, from which the A_j can be successively calculated.

$$A_3 = \mu_{03} - 3\mu_{11}$$

$$A_4 = \mu_{04} - 6\mu_{12} + 4A_3\mu_{11} + 3\mu_{20}$$

$$A_5 = \mu_{05} - 10\mu_{13} + 10A_3\mu_{12} + 15\mu_{21} - 5A_4\mu_{11} - 10A_3\mu_{20}$$

$$\begin{aligned} A_6 = \mu_{06} &- 15\mu_{14} + 20A_3\mu_{13} + 45\mu_{22} - 15A_4\mu_{12} - 60A_3\mu_{21} \\ &+ 6A_5\mu_{11} - 15\mu_{30} + 10A_3^2\mu_{20} + 15A_4\mu_{20} \end{aligned}$$

$$\begin{aligned} A_7 = \mu_{07} &- 21\mu_{15} + 35A_3\mu_{14} + 105\mu_{23} - 35A_4\mu_{13} - 210A_3\mu_{22} \\ &+ 21A_5\mu_{12} + 105\mu_{31} + 70A_3^2\mu_{21} + 105A_4\mu_{21} - 7A_6\mu_{11} \\ &- 21A_5\mu_{20} + 105A_3\mu_{30} - 35A_3A_4\mu_{20} \end{aligned}$$

$$\begin{aligned} A_8 = \mu_{08} &- 28\mu_{16} + 56A_3\mu_{15} + 210\mu_{24} - 70A_4\mu_{14} - 560A_3\mu_{23} \\ &+ 56A_5\mu_{13} - 420\mu_{32} + 420A_4\mu_{22} + 280A_3^2\mu_{22} - 28A_6\mu_{12} \\ &+ 840A_3\mu_{31} - 168A_5\mu_{21} - 280A_3A_4\mu_{21} + 8A_7\mu_{11} \\ &+ 105\mu_{40} - 210A_4\mu_{25} - 280A_3^2\mu_{30} + 28A_6\mu_{20} \\ &+ 28A_6\mu_{20} + 56A_3A_5\mu_{20} + 35A_4^2\mu_{20} . \end{aligned}$$

We have found that the numerical calculation of these $\{A_n\}$-coefficients is in no way facilitated by solving the preceding equations for explicit representations of the $\{A_n\}$ in terms of the product moments. A computer program is easily prepared to make systematic use of the successive equations for $A_3, A_4, \ldots, A_8$ if they are needed.

The coefficients $\{B_n\}$ of Theorem 3.1 are far less important than the $\{A_n\}$ since they merely provide a second-order improvement in the formulae for the cumulants of W(t). Unfortunately, they are much more troublesome to calculate than the $\{A_n\}$, and no obvious simplifying tricks seem available. If we write

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{B_n}{n!} (i\theta)^n \tag{4.11}$$

then it follows from (3.11) and (3.3) that

$$e^{\phi(\theta)} = \frac{1 - G(0,\theta)}{z_1(\theta) K(\theta)}, \text{ say,} \tag{4.12}$$

where

$$K(\theta) = \left[\frac{\partial}{\partial s} G(s,\theta)\right]_{s=z_1(\theta)} . \tag{4.13}$$

Let us write

$$\mu_{or}(\theta) = [(-\frac{\partial}{\partial s})^r G(s,\theta)]_{s=0} = \sum_{n=0}^{\infty} \frac{\mu_{rn}}{n!} (i\theta)^n . \tag{4.14}$$

Then

$$\frac{1 - G(0,\theta)}{-z_1(\theta)} = \frac{G(z_1(\theta),\theta) - G(0,\theta)}{-z_1(\theta)} = \sum_{r=1}^{\infty} \frac{[-z_1(\theta)]^{r-1}}{r!} \mu_{or}(\theta) \tag{4.15}$$

In order to avoid too much complexity we shall, at this point, suppose $\mu_{01} = 0$. This represents a genuine loss of generality, not trivially overcome as a similar maneuver was for the case of the $\{A_n\}$. However, our method of attack also becomes clearer to follow. We shall also, with no further loss of generality, assume the X and Y scales have been adjusted to make $\mu_{10} = \mu_{02} = 1$.

If we were only interested in terms up to those in $(i\theta)^4$, the right-hand side of (4.14) equals, to terms of that degree,

$$\{1 + i\theta\mu_{11} + \frac{(i\theta)^2}{2!}\mu_{12} + \frac{(i\theta)^3}{3!}\mu_{13} + \frac{(i\theta)^4}{4!}\mu_{14}\}$$

$$- \frac{1}{2!}\{\mu_{20} + i\theta\mu_{21} + \frac{(i\theta)^2}{2!}\mu_{22}\}\{\frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!}A_3 + \frac{(i\theta)^4}{4!}A_4\}$$

$$+ \frac{1}{3!}\{\frac{(i\theta)^2}{2!}\}^2\mu_{30}.$$

Let us write $C_m/m!$ for the coefficient of $(i\theta)^m$ in this expression. Then we find

$$C_0 = 1$$

$$C_1 = \mu_{11}$$

$$C_2 = \mu_{12} - \frac{1}{2}\mu_{20} \tag{4.16}$$

$$C_3 = \mu_{13} - \frac{1}{2}\mu_{20}A_3 - \frac{3}{2}\mu_{21}$$

$$C_4 = \mu_{14} - \frac{1}{2}\mu_{20}A_4 - 2\mu_{21}A_3 - 3\mu_{22} + \mu_{30}$$

From the expansion

$$\frac{1 - G(0,\theta)}{-z_1(\theta)} = 1 + C_1(i\theta) + \frac{C_2}{2!}(i\theta)^2 + \ldots + \frac{C_4}{4!}(i\theta)^4 + O(\theta^5)$$

we can discover the coefficients C_1', C_2', ..., say, in the logarithmic expansion

$$\log \frac{1 - G(0,\theta)}{-z_1(\theta)} = C_1'(i\theta) + \frac{C_2'}{2!}(i\theta)^2 + \ldots + \frac{C_4'}{4!}(i\theta)^4 + O(\theta^5) .$$

The easiest procedure for the calculation of the C_j' is to make use of the tables provided by Kendall and Stuart for the calculation of cumulants from moments (Kendall and Stuart, 1958, p. 70). In the present instance we find:

$$C_1' = \mu_{11}$$

$$C_2' = \mu_{12} - \frac{1}{2}\mu_{20} - \mu_{11}^2$$

$$C_3' = \mu_{13} - \frac{1}{2}\mu_{20}A_3 - \frac{3}{2}\mu_{21} - 3\mu_{11}\mu_{12} + \frac{3}{2}\mu_{11}\mu_{20} + 2\mu_{11}^3$$

$$\begin{aligned} C_4' = \mu_{14} &- \frac{1}{2}\mu_{20}A_4 - 2\mu_{21}A_3 + 2\mu_{11}\mu_{20}A_3 - 3\mu_{22} + \mu_{30} - 4\mu_{11}\mu_{13} \\ &+ 6\mu_{11}\mu_{21} - 3\mu_{12}^2 + 3\mu_{12}\mu_{20} - \frac{3}{4}\mu_{20}^2 + 12\mu_{11}^2\mu_{12} \\ &- 6\mu_{11}^2\mu_{20} - 6\mu_{11}^4 . \end{aligned}$$

A glance at (4.12) shows that we also need an expansion of $\log\{-K(\theta)\}$ as a power series in θ, if we are to determine the $\{B_n\}$.

Let us set

$$\log\{-K(\theta)\} = C_1''(i\theta) + \frac{C_2''}{2!}(i\theta)^2 + \ldots$$

Then, from (4.13), it follows that

$$K(\theta) = \left[\frac{\partial}{\partial s}\left\{\sum_{r=0}^{\infty}\frac{s^r}{r!}\left[\frac{\partial^r}{\partial\sigma^r}G(\sigma,\theta)\right]_{\sigma=0}\right]_{s=z_1(\theta)}$$

$$= \sum_{r=0}^{\infty}\frac{[z_1(\theta)]^r}{r!}\left[\frac{\partial^{r+1}}{\partial\sigma^{r+1}}G(\sigma,\theta)\right]_{\sigma=0}.$$

But

$$(-1)^{r+1}\left[\frac{\partial^{r+1}}{\partial\sigma^{r+1}}G(\sigma,\theta)\right]_{\sigma=0} = \sum_{j=0}^{\infty}\frac{(i\theta)^j}{j!}\mu_{r+1,j}$$

so that

$$-K(\theta) = \sum_{r=0}^{\infty}\frac{[-z_1(\theta)]^r}{r!}\left\{\sum_{j=0}^{\infty}\frac{(i\theta)^j}{j!}\mu_{r+1,j}\right\}$$

$$= \sum_{r=1}^{\infty}\frac{[-z_1(\theta)]^{r-1}}{(r-1)!}\left\{\sum_{j=0}^{\infty}\frac{(i\theta)^j}{j!}\mu_{rj}\right\}.$$

On comparing this expression with (4.14) and (4.15) we see that formulae for C''_1, C''_2, ..., can be obtained from those for C'_1, C'_2, ..., by replacing every product moment $\mu_{\alpha\beta}$ at every appearance in (4.16) by $\alpha\mu_{\alpha\beta}$. Thus, for example, we can infer

$$C''_3 = \mu_{13} - \mu_{20}A_3 - 3\mu_{21} - 3\mu_{11}\mu_{12} + 3\mu_{11}\mu_{20} + 2\mu_{11}^2 .$$

From (4.11) and (4.12) we see that

$$B_r = C'_r - C''_r , \quad r = 1,2, \ldots .$$

Thus we can deduce some easy rules for obtaining the $\{B_n\}$ from the formulae (4.16) for the $\{C_n'\}$.

(i) Eliminate terms in C_n' involving only product moments of the type $\mu_{1\beta}$.

(ii) A term in C_n' involving only one product moment $\mu_{\alpha\beta}$ with $\alpha \geq 2$ (possibly multiplied by one or more $\mu_{1\beta}$); gets multiplied by $-(\alpha - 1)$.

(iii) A term in C_n' involving two product moments $\mu_{\alpha\beta}$, $\mu_{\gamma\delta}$, say, with $\alpha \geq 2$ and $\gamma \geq \delta$, gets multiplied by $-(\alpha\gamma - 1)$.

It should be obvious how these rules arise and how they may, when necessary, be extended. They lead to the following results: -

$$B_1 = 0$$

$$B_2 = \frac{1}{2}\mu_{20}$$

$$B_3 = \frac{1}{2}\mu_{20}A_3 + \frac{3}{2}\mu_{21} - \frac{3}{2}\mu_{11}\mu_{20}$$

$$B_4 = \frac{1}{2}\mu_{20}A_4 + 2\mu_{21}A_3 - 2\mu_{11}\mu_{20}A_3 + 3\mu_{22} - 2\mu_{30} - 6\mu_{11}\mu_{21} - 3\mu_{12}\mu_{20} + \frac{9}{4}\mu_{20}^2 + 6\mu_{11}^2\mu_{20}$$

$$\begin{aligned} B_5 = {} & 5\mu_{23} + 5A_3\mu_{22} + \frac{5}{2}A_4\mu_{21} + \frac{1}{2}A_5\mu_{20} - 10\mu_{31} - \frac{20}{3}A_3\mu_{30} \\ & - 15\mu_{11}\mu_{22} - 10A_3\mu_{11}\mu_{21} - \frac{5}{2}A_4\mu_{11}\mu_{20} + 10\mu_{11}\mu_{30} \\ & - 15\mu_{12}\mu_{21} - 5A_3\mu_{12}\mu_{20} - 5\mu_{20}\mu_{13} + \frac{45}{2}\mu_{20}\mu_{21} + \frac{15}{2}A_3\mu_{20}^2 \\ & + 30\mu_{11}^2\mu_{21} + 10A_3\mu_{11}^2\mu_{20} + 30\mu_{11}\mu_{12}\mu_{20} - \frac{45}{2}\mu_{11}\mu_{20}^2 - 30\mu_{11}^3\mu_{20} \end{aligned}$$

We close this section with the comment, promised earlier, of the effect of the assumption $\nu_1 = 0$. The formulae given for $\{A_n\}$ and $\{B_n\}$ are correct as they stand when $\nu_1 = 0$. If $\nu_1 \neq 0$ then the correct values for $\{A_n\}_{n\geq 2}$ can be obtained if every product moment μ_{rs} used in the formulae for the $\{A_n\}$ is replaced by $\tilde{\mu}_{rs} = EX_1^r(Y_1-\nu_1 X_1)^s$; it should be borne in mind that the time and "y" scales have been adjusted to make $\mu_1 = EX_1 = 1$ and $E(Y_1-\nu_1 X_1)^2 = 1$, something which can be achieved with no loss of generality.

If $\nu_1 \neq 0$ we have provided no results about the $\{B_n\}$; it is not an impossible task to obtain them by the methods we have used earlier and B_1, B_2, B_3, B_4 should not be too complicated. For the present, however, we have shunned their calculation.

§5. Checks on the calculation.

It is desirable to devise checks on the formulae obtained in the previous section since there is plainly ample opportunity for slips in the calculation of the $\{A_n\}$ and $\{B_n\}$ for higher values of n.

If the $\{X_n\}$ are iid with pdf e^{-x} and the $\{Y_n\}$ are *independent* of the $\{X_n\}$ then it is easy to see that W(t) is a compound Poisson process and, indeed, that

$$(5.1) \qquad E\, e^{i\theta W(t)} = H(\theta)\, \exp\text{-}\{1 - H(\theta)\}t\ .$$

In this case it is clear that

$$(5.2) \qquad A_n = \nu_n \qquad n = 1,2,\dots\ .$$

Furthermore, if we write $\{K_n\}$ for the successive *cumulants* of H, then (5.1) makes it plain that

$$(5.3) \qquad B_n = K_n \qquad n = 1,2,\dots\ .$$

Thus, if we set $\nu_1 = 0$, and

$$(5.4) \qquad \mu_{\alpha\beta} = (\alpha!)\nu_\beta$$

for all relevant values of α and β, then the formulae of §4 for $\{A_n\}$ and $\{B_n\}$ should agree with (5.2) and (5.3). We call this the *Compound Poisson Check*.

All our formulae pass this check satisfactorily. Unfortunately, because every product-moment like $\mu_{\alpha 1}$ must be set to zero (because X_j and Y_j are independent and $\nu_1 = 0$), this check give no reassurance at all that terms involving such product-moments have correct coefficients.

A more elaborate test, which we call the *Double Poisson Check* is as follows. Let the X_n be distributed as for the Compound Poisson Check, but let $Y_n = 1 - X_n$. For this model one has

$$W(t) = N(t) + 1 - S_{N(t)+1} \tag{5.5}$$
$$= N(t) + 1 - t - S(t),$$

where N(t) is a random variable with a Poisson distribution, $\mathcal{E}\, N(t) = t$, and $\xi(t)$ is a non-negative random variable, independent of N(t), with pdf e^{-x}. Thus (5.5) gives

$$\mathcal{E}\, e^{i\theta W(t)} = \frac{e^{i\theta}}{1 + i\theta} \exp t(e^{i\theta} - 1 - i\theta) \tag{5.6}$$

From (5.6) we see that

$$A_n = 1 \qquad n = 2,3,\ldots \tag{5.7}$$

and

$$(-1)^n B_n = (n - 1)! \qquad n = 2,3,\ldots \tag{5.8}$$

For this check it is necessary to calculate a fair number of the moments

$$\mu_{\alpha\beta} = \int_0^\infty e^{-x} x^\alpha (1 - x)^\beta dsc. \tag{5.9}$$

A table of the necessary values is given below, since their computation is tedious once the values of α and β get moderately large.

Values of product-moments $\mu_{\alpha\beta}$ for Double Poisson check

β \ α	0	1	2	3	4
0	1	1	2	6	24
1	0	-1	-4	-18	-96
2	1	3	14	78	504
3	-2	-11	-64	-426	-3,216
4	9	53	362	2,790	24,024
5	-44	-309	-2,428	-21,234	--
6	265	2,119	18,806	--	--
7	-1,854	-16,687	--	--	--
8	14,833	--	--	--	--

When the values of the product-moments $\mu_{\alpha\beta}$ given in this table are used in the formulae for $\{A_n\}$ and $\{B_n\}$ of §4, the values obtained are in full agreement with those predicted by (5.7) and (5.8). This Double Poisson Check involves all coefficients involved in our formulae and the fact that they pass this check strongly suggests that none of these formulae contain an error.

§6. Brief comments on Class B Cumulative Processes.

Suppose W(t) to be a Class B cumulative process instead of Class A as we have been supposing throughout this paper. The arguments showing that W(t) has cumulants of the asymptotically linear form exemplified in Theorem 3.1 will go through much as for the Class A case; the remainder terms will have the same properties. Furthermore, the calculations of §3 leading to (3.3) can be imitated, and will lead to the result

$$M_\theta^0(s) = \frac{1 - F(s)}{s[1 - G(s,\theta)]} . \tag{6.1}$$

It is clear from this, as it should be intuitively in any case, that the coefficients $\{A_n\}$ will be exactly the same as for the Class A process, obtained by expanding the root $z_1(\theta)$ of the equation $G(z_1(\theta),\theta) = 1$. There will be a difference, however, in the coefficients $\{B_n\}$. If we once again set

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{B_n}{n!} (i\theta)^n$$

then it can be shown from (6.1) that

$$e^{\phi(\theta)} = - \frac{1 - F(z_1(\theta))}{z_1(\theta)K(\theta)} , \tag{6.2}$$

where $K(\theta)$ is as given in (4.13). This result (6.2) should be compared with (4.12) obtained for the case of the Class A process.

Nothing but the need for careful and tedious computation is needed to extract from (6.2) the desired coefficients $\{B_n\}$, but we shall not proceed further along such lines in the present report.

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Let $\{(X_n,Y_n)\}$, n=1,2,..., be a sequence of iid 2-vectors such that $P\{X_n > 0\} = 1$ and write $S_o = 0$ and $S_n = X_1 + X_2 + \ldots + X_n$, $n \geq 1$. Let N(t) be the maximum k such that $S_k \leq t$. A Class A cumulative process is

$$W(t) = \sum_{1}^{N(t)+1} Y_j .$$

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20. (continued)

Let $K_m(t)$ be the m^{th} cumulant of $W(t)$. Under general conditions involving the existence of various product moments and the hypothesis that the $\{X_n\}$ are non-lattice, it is shown that there exist constants A_m and B_m such that, as $t \to \infty$,

$$K_m(t) = A_m t + B_m + R_m(t) .$$

Here $R_m(t) \to 0$ as $t \to \infty$, and various things can be said about $R_m(t)$, depending on hypothesis on the joint df of (X_n, Y_n). Expressions are obtained for A_m, $m = 1,2,\ldots,8$ and B_m, $m = 1,2,\ldots,6$ in terms of the product moments $\mu_{\alpha\beta} = E\, X_n^{\alpha} Y_n^{\beta}$, and methods of checking these complicated formulae are discussed.